RESEARCH STATEMENT

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I am an algebraic geometer interested in rational curves, hypersurfaces, algebraic hyperbolicity, and birational geometry. My research concentrates on the algebraic hyperbolicity of hypersurfaces in homogeneous varieties and normal bundles of rational curves on varieties in projective space.

A complex projective variety X is *(Brody) hyperbolic* when it admits no nonconstant holomorphic map $\mathbb{C} \to X$, that is, when it contains no entire curves. In dimension one, hyperbolic curves are those with genus greater or equal to 2. In higher dimensions, it is an important and challenging problem to characterize hyperbolic varieties, and famous conjectures have been formulated in this direction:

- Kobayashi conjectured [Kob70] that a very general hypersurface of degree $d \geq 2n$ in $\mathbb{P}^n$ is hyperbolic.
- Lang conjectured [Lan86] that a projective algebraic variety X is hyperbolic if and only if every subvariety of X is of general type.
- The Green-Griffiths-Lang Conjecture [GG80] says that a smooth projective variety X of general type contains a proper subvariety $S \subsetneq X$ containing all entire curves of X.

From the arithmetic perspective, the Mordell-Faltings Theorem states that a smooth curve C of genus $g \geq 2$ defined over a number field K has only finitely many rational points over any finite field extension L of K. In higher dimensions, Lang conjectured [Lan86]:

- A complex projective variety X is hyperbolic if and only if it contains only finitely many rational points over any finite field extension.
- If X is of general type, then there exists a proper algebraic subset Z of X such that, over any finite extension L of K, the number of L-rational points of $X \setminus Z$ is finite.

Thus, understanding hyperbolicity is a fundamental step in understanding the geometry and arithmetic of varieties of general type. Even though these conjectures remain open in general, progress has recently been made in particular cases. Demailly [Dem97] introduces algebraic hyperbolicity as an algebraic analogue for Brody hyperbolicity: we say X is *algebraically hyperbolic* if there exists an ample divisor H and a real number $\epsilon > 0$ such that the geometric genus $g(C)$ and the degree of any integral curve $C \subset X$ satisfy the inequality

$$2g(C) - 2 \geq \epsilon \deg_H(C).$$

In particular, algebraically hyperbolic varieties do not contain any rational or elliptic curves. Every hyperbolic variety is algebraically hyperbolic, and Demailly conjectured that the converse holds.

The algebraic hyperbolicity of very general hypersurfaces in projective space is almost completely classified by [Cle86a; Ein88; Ein91; Voi96; Pac03; Pac04; CR04; CR23; Yeo22]. By building on their techniques, I extended the classification to the much more general case of homogeneous varieties, thus obtaining a bound for the hyperbolicity in plenty of open cases, including Grassmannians, flag varieties, and their products [Mio23a]:

Theorem 0.1. *[Mio23a] I give an almost optimal bound for the degree d such that a very general degree d hypersurface of a homogeneous variety is algebraically hyperbolic. In particular, I obtain explicit bounds for the degrees when very general hypersurfaces in Grassmannians and flag varieties are algebraically hyperbolic.*

The explicit bounds are given in Section 1, together with a more complete description of my research done in [Mio23a]. Additionally, I propose future research directions in the study of varieties of general type.

A *rational curve* in $\mathbb{P}^n$ is a non-constant polynomial map $f : \mathbb{P}^1 \to \mathbb{P}^n$. In general, it is a curve birationally equivalent to $\mathbb{P}^1$, a genus 0 curve. When it is smooth, it is isomorphic to $\mathbb{P}^1$. Rational curves are among the most classical objects in Algebraic Geometry, and they play a central role in understanding the birational and arithmetic geometry of projective varieties. If on algebraic hyperbolicity we study varieties with no rational curves, here we look at varieties with many rational curves. By studying the geometry of these rational curves, we can better understand the intrinsic geometry of varieties. This area of study contains many basic but deep questions that are still open:

- The Clemens Conjecture [Cle86b] says that a general quintic threefold in $\mathbb{P}^4$ contains a nonzero finite number of smooth rational curves of any given degree. Mirror Symmetry gives a prediction for the number of curves.
- Let X be a smooth degree d hypersurface in $\mathbb{P}^n$ with $d \leq n$. The Debarre-de Jong Conjecture [EH16] says that the space of lines on X has the expected dimension $2n-3-d$.
- Let X be a general degree d hypersurface in $\mathbb{P}^n$ with $(n,d) \neq (3,4)$. Coskun-Harris-Starr Conjecture [HRS04] suggests that the space of degree e rational curves in X has dimension equal to $\max\{-1, e(n-d+1)+n-4\}$.
- Voisin conjectured [Voi03] that on a very general degree d hypersurface $X \subset \mathbb{P}^n$ with $d > n+1$, the degrees of rational curves are bounded.

These problems centralize a very active research subject and have recently seen substantial progress. For example, see [Kat86; JK95; JK97; Cot05; Cot12] for Clemens' Conjecture, [BK98; Beh06; LT10b; LT10a; Beh14; BR21] for Debarre-de Jong Conjecture, [HRS04; BK13; RY19] for Coskun-Harris-Starr Conjecture, and [Cle86a; Ein88; Voi96; Voi98; Pac03; RY20; Abe23] for Voisin's Conjecture.

The normal bundle of a rational curve controls the local structure of the space of rational curves in a variety. By understanding the normal bundles, we understand how the rational curves deform, a tool at the core of the study of these rational curves. The normal bundle $N_{C/X}$ of a rational curve C on a variety X splits as a direct sum of line bundles $N_{C/X} \cong \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$. The collection of integers a_i is called the splitting type of $N_{C/X}$. An important and difficult problem is to classify what splitting types of $N_{C/X}$ are achieved for some given curve C or variety X.

In [Mio23b], I completely solve this problem for rational normal curves on hypersurfaces in $\mathbb{P}^n$: I find all splitting types of $N_{C/X}$ achieved for some degree d hypersurface X, and create a method to construct explicit examples of hypersurfaces for each splitting type:

Theorem 0.2. *[Mio23b, Theorems 3.1 and 4.3] Let C be a rational normal curve of degree $e \leq n$ in $\mathbb{P}^n$. We find all possible collections $\{a_i\}_{i=1}^{n-2}$ for which there exists a degree $d \geq 2$ hypersurface X containing C with splitting type $N_{C/X} \cong \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^1}(a_i)$. In addition, for each possible splitting type, we obtain an explicit example of such a hypersurface X.*

For a given splitting type $E_{\vec{a}} \cong \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^1}(a_i)$, it was expected that the locus $\Sigma_{\vec{a}}$ of degree d hypersurfaces in $\mathbb{P}^n$ containing C with $N_{C/X} \cong E_{\vec{a}}$ had codimension $h^1(\mathbb{P}^1, \mathcal{E}nd(E_{\vec{a}})) = \sum_{\{i,j|a_i-a_j \leq -2\}}(a_j-a_i-1)$. When $d \geq 3$, my method developed in [Mio23b] also allows us to obtain a very good description of $\Sigma_{\vec{a}}$ and compute its dimension. In particular, we show that its actual codimension does not always coincide with the expected codimension, and the difference can get arbitrarily large as n grows:

Theorem 0.3. *[Mio23b, Theorem 3.6] Given a possible splitting type $E_{\vec{a}}$, let z be the number of terms $\mathcal{O}_{\mathbb{P}^1}(e+2)$ in $E_{\vec{a}}$. Then the locus of degree $d \geq 3$ hypersurfaces containing C with $N_{C/X} \cong E_{\vec{a}}$ is irreducible and smooth of codimension $h^1(\mathbb{P}^1, \mathcal{E}nd(E_{\vec{a}})) - (n-e)z$.*

In Section 2, I present a more complete description of my results and discuss further research problems on rational curves.

1. Algebraic hyperbolicity of very general hypersurfaces

A lot of progress has been made in the study of algebraic hyperbolicity of very general hypersurfaces in $\mathbb{P}^n$. For $n = 3$, Xu [Xu94] proved that very general hypersurfaces $X \subset \mathbb{P}^3$ of degree $d \geq 6$ are algebraically hyperbolic. This was improved by Coskun and Riedl [CR19a], who showed that also a very general quintic surface is algebraically hyperbolic. Since surfaces of degree at most four contain rational curves and cannot be algebraically hyperbolic, the classification for very general hypersurfaces in $\mathbb{P}^3$ is complete. For $n \geq 4$, Clemens and Ein [Cle86a; Ein88] proved that X is algebraically hyperbolic when $d \geq 2n$. This was improved to $d \geq 2n - 1$ by Voisin [Voi96; Voi98]. Pacienza [Pac03; Pac04] and Clemens and Ran [CR04] proved it for $d \geq 2n - 2$ and $n \geq 6$. Yeong [Yeo22] did the octic 4-fold case and improved the result to $d \geq 2n - 2$ and $n \geq 5$. If $d \leq 2n - 3$, X contains lines, thus it is not algebraically hyperbolic. Hence, the only remaining case in $\mathbb{P}^n$ is for sextic threefolds. We note that the problem becomes fundamentally harder as the degree d decreases, so obtaining the exact bound is especially challenging.

Haase and Ilten [HI21] started to classify algebraically hyperbolic surfaces in toric threefolds. Coskun and Riedl completed their classification in [CR23]. By applying their techniques to products of projective spaces, Yeong [Yeo22] almost completely classified algebraically hyperbolic hypersurfaces in $\mathbb{P}^m \times \mathbb{P}^n$, except for some degrees in $\mathbb{P}^3 \times \mathbb{P}^1$.

By building on their techniques, I was able to work out the classification of algebraically hyperbolic hypersurfaces to the much more general case of homogeneous varieties. I worked out a broad setting where I could prove the general theorem that gives an almost optimal bound for the degree. The theorem can then be applied to get an explicit bound for the degree for which a very general hypersurface in Grassmannians, flag varieties, or their products is algebraically hyperbolic.

First, I introduce the general theorem in the following setting: let A be a smooth complex variety embedded into a product of projective spaces $\mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_m}$ so that A is projectively normal. Suppose its Picard group is generated by the pull-backs of the hyperplanes classes $H_1, \ldots, H_m$, and assume that A admits a transitive group action by an algebraic group G. Write the class of the canonical divisor of A as $K_A = \sum_{i=1}^m a_i H_i$. Let $\mathcal{E}$ be the line bundle of $\sum_{i=1}^m d_i H_i$ for $d_1, \ldots, d_m > 0$, invariant under G, and assume that $H_1, \ldots, H_m$ is a collection of section-dominating line bundles for $\mathcal{E}$. Let X be a very general section of $\mathcal{E}$.

Theorem 1.1. *[Mio23a, Theorem 4.3] Suppose that* $\dim A \geq 4$.

If $d_i \geq \dim A - a_i - 2$ *for all* $1 \leq i \leq m$*, then* X *is algebraically hyperbolic.*

If $d_i \leq \dim A - a_i - 4$*, then* X *contains* H_i*-lines. Therefore, if* $d_i \leq \dim A - a_i - 4$ *for some* $1 \leq i \leq m$*, then* X *is not algebraically hyperbolic.*

We then apply the theorem to examples of homogeneous varieties, obtaining the following bounds:

Theorem 1.2. *[Mio23a, Section 5]*

(1) ***Grassmannians*** $A = G(k, n)$

If $\dim A = k(n-k) \geq 4$ *and* $d \geq k(n-k)+n-2$*, then a very general degree* d *hypersurface of* $G(k, n)$ *is algebraically hyperbolic.*

If $d \leq k(n-k) + n - 4$*, then a general degree* d *hypersurface of* $G(k, n)$ *contains a line. In particular, it is not algebraically hyperbolic.*

(2) ***Products of Grassmannians*** $A = \prod_{i=1}^m G(k_i, n_i)$

If $\dim A = \sum_{i=1}^m k_i(n_i - k_i) \geq 4$ *and* $d_i \geq \left(\sum_{j=1}^m k_j(n_j - k_j)\right) + n_i - 2$*, for all* $1 \leq i \leq m$*, then a very general hypersurface of* $\prod_{i=1}^m G(k_i, n_i)$ *of degree* $(d_1, \ldots, d_m)$ *is algebraically hyperbolic.*

If for some i*,* $d_i \leq \left(\sum_{j=1}^m k_j(n_j - k_j)\right) + n_i - 4$*, then a general hypersurface of degree* $(d_1, \ldots, d_m)$ *of* $\prod_{i=1}^m G(k_i, n_i)$ *contains a line. In particular, it is not algebraically hyperbolic.*

(3) ***Orthogonal Grassmannians*** $A = OG(k,n)$

If $\dim OG(k,n) = \frac{k(2n-3k-1)}{2} \geq 4$ *and* $d \geq \frac{k(2n-3k-1)}{2} + n - 3k - 1$, *then a very general hypersurface of* $OG(k,n)$ *of degree* d *is algebraically hyperbolic.*

If $d \leq \frac{k(2n-3k-1)}{2} + n - 3k - 3$, *then a general hypersurface of degree* d *of* $OG(k,n)$ *contains a line. In particular, it is not algebraically hyperbolic.*

(4) ***Symplectic Grassmannians*** $A = SG(k,n)$

If $\dim A = \frac{k(2n-3k+1)}{2} \geq 4$ *and* $d \geq \frac{k(2n-3k+1)}{2} + n + 3k$, *then a very general degree* d *hypersurface of* $SG(k,n)$ *is algebraically hyperbolic.*

If $d \leq \frac{k(2n-3k+1)}{2} + n + 3k - 2$, *then a general degree* d *hypersurface of* $SG(k,n)$ *contains a line. In particular, it is not algebraically hyperbolic.*

(5) ***Flag Varieties*** $A = F(k_1, \dots, k_m; n)$

If $\dim A = \sum_{i=0}^{m} k_{i+1}(k_{i+1} - k_i) \geq 4$ *and* $d_i \geq \left(\sum_{j=0}^{m} k_{j+1}(k_{j+1} - k_j)\right) + k_{i+1} - k_{i-1} - 2$ *for all* $1 \leq i \leq m$, *then a very general hypersurface of* $F(k_1, \dots, k_m; n)$ *of degree* $(d_1, \dots, d_m)$ *is algebraically hyperbolic.*

If $d_i \leq \left(\sum_{j=0}^{m} k_{j+1}(k_{j+1} - k_j)\right) + k_{i+1} - k_{i-1} - 4$ *for some* i, *then a general hypersurface of* $F(k_1, \dots, k_m; n)$ *of degree* $(d_1, \dots, d_m)$ *contains a line. In particular, it is not algebraically hyperbolic.*

To prove this theorem, we start from a universal family $\mathcal{X}$ of very general degree d hypersurfaces in A, and a universal curve $h : \mathcal{Y} \to \mathcal{X}$. Let π be the projection $\mathcal{X} \to A$. We define the vertical tangent sheaves $T_{\mathcal{X}/A}$ and $T_{\mathcal{Y}/A}$ as the kernels in the short exact sequences:

$$0 \longrightarrow T_{\mathcal{X}/A} \longrightarrow T_{\mathcal{X}} \longrightarrow \pi_2^* T_A \longrightarrow 0,$$

and

$$0 \longrightarrow T_{\mathcal{Y}/A} \longrightarrow T_{\mathcal{Y}} \longrightarrow h^* \pi_2^* T_A \longrightarrow 0.$$

We can then show that $T_{\mathcal{X}/A}$ is isomorphic to the Lazarsfeld-Mukai bundle $\pi^* M_{\mathcal{E}}$, and that the normal bundle $N_{h/\mathcal{X}}$ is the cokernel of the map $T_{\mathcal{Y}/A} \to T_{\mathcal{X}/A}$. For simplicity, denote a general degree d hypersurface of $\mathcal{X}$ by X and by $C \subset X$ its genus g curve. Thus, we get a surjection $M_{\mathcal{E}}|_C \to N_{C/X}$. On the other hand, the sections $H_1, \dots, H_m$ induce a surjection $\bigoplus_{i=1}^{m} M_{H_i}^{\oplus s_i} \to M_{\mathcal{E}}$ for some integers s_i. These then give the surjection

$$\beta : \bigoplus_{i=1}^{m} M_{H_i}^{\oplus s_i}|_C \to N_{C/X},$$

which can be used to obtain a lower bound for the geometric genus of C:

$$2g(C) - 2 \geq K_X \cdot C - \sum_{i=1}^{m} s_i \deg H_i|_C.$$

The above bound suffices to prove the theorem when every $s_i < \dim A - 2 = \operatorname{rk} N_{C/X}$. Otherwise, we construct a surface scroll containing C, which is used to further improve the bound.

1.1. **Further research problems.** I am interested in understanding the geometry of projective varieties of general type. The hyperbolicity and Green-Griffiths-Lang conjectures are important steps in this direction and have been the object of numerous works.

For instance, results due to Clemens [Cle86a], Ein [Ein88; Ein91], Voisin [Voi96] and Pacienza [Pac04] state that every subvariety Y of a general algebraic hypersurface $X \subset \mathbb{P}^n$ of degree $d \geq 2n-2$ is of general type for $n \geq 5$. The same bound would hold for hyperbolicity as a consequence of the Green-Griffiths-Lang Conjecture. In a deep statement, Diverio, Merker, and Rousseau [SR10] confirm the Green-Griffiths-Lang Conjecture for a large degree $d \geq 2^{n^5}$ by applying a jet differential bundles method.

I plan to contribute to the study of these conjectures, which remain unsolved in general. To better understand the problem, it is helpful to approach particular examples and test different scenarios. Some proposed directions are outlined below.

1.1.1. *Obtaining the optimal bound.* Theorem 1.1 naturally raises the question: what can we say about the case $d_i = \dim A - a_i - 3$?

The answer to this question is not so simple, and it might depend on the dimension of A. When $\dim A = 4$, Theorem 1.1 bound is sharp as we can show that a very general hypersurface of degree $(4, 4)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ contains an elliptic curve. For $A = \mathbb{P}^4$, it corresponds to the case of sextic threefolds, which remains open. In addition, [CR23] shows that in $A = \mathbb{P}^2 \times \mathbb{P}^1$, X is algebraically hyperbolic if and only if $d_1 \geq 4$ and $d_2 \geq 3$, or $d_1 \geq 5$ and $d_2 = 2$, suggesting that low degrees can be compensated by higher degrees in other factors.

However, for $A = \mathbb{P}^m \times \mathbb{P}^n$ and $\dim A \geq 5$, the bound $d_i \geq \dim A - a_i - 3$ for all i does imply algebraic hyperbolicity, as proved in [Yeo22]. So, we conjecture:

Conjecture 1.3. *[Mio23a, Question 4.6] If $\dim A \geq 5$ and $d_i \geq \dim A - a_i - 3$ for all i, then X is algebraically hyperbolic.*

By applying the proof of Theorem 1.1 to this bound, we are only left with the case when $s_1 = \dim A - 2$ and $s_2, \ldots, s_m = 0$, that is, when there is a surjection

$$M_{H_1}^{\oplus \dim A - 2} \to N_{C/X}.$$

In the analogous case for $\mathbb{P}^m \times \mathbb{P}^n$, Yeong used an osculation locus argument to derive the result in [Yeo22]. According to it, through every point of C passes a line that meets X in exactly one point, thus C is contained in what is called the osculation locus of X. Following Voisin and Pacienza's work, it can be shown that this locus is a curve of general type.

In our case, it can be shown that there is a line through every point of C via the scroll construction. I am confident I can find a similar locus for X in homogeneous varieties and show it to be of general type.

1.1.2. *Toric varieties and other applications.* The initial setup of Theorem 1.1 can also be defined when A is not homogeneous, but contains a Zariski-open homogeneous subset. This allowed Coskun and Riedl [CR23] to classify algebraically hyperbolic surfaces X in the cases when $A =$ a Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$, $A = \mathbb{P}^3$ blown up at a single point, and $A =$ a weighted projective space $\mathbb{P}(1, 1, 1, n)$.

By building up on my generalization done in Theorem 1.1, I believe I can extend my result to varieties with a open homogeneous subset. We could then use it to classify algebraic hyperbolicity in higher dimensional toric varieties. For example, if A is the blowup of $\mathbb{P}^n$ at a linear subspace $\Lambda \cong \mathbb{P}^k$, we are in a setting similar to Theorem 1.1: its exceptional divisor is $E \cong \mathbb{P}^k \times \mathbb{P}^{n-k-1}$, its Picard group is generated by E and $H - E$, where H is the pullback of the hyperplane class in $\mathbb{P}^n$, and we get a map to the product of projective spaces $A \to \mathbb{P}^{n-k-1} \times \mathbb{P}^n$. More generally:

Question 1.4. Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_j$ be linear subspaces of $\mathbb{P}^n$, and let A be the blowup of $\mathbb{P}^n$ at $\Lambda_1, \Lambda_2, \ldots, \Lambda_j$. For what degrees is the very general hypersurface in A algebraically hyperbolic?

1.1.3. *Complete intersections.* Much of the mentioned work on algebraic hyperbolicity starts from the techniques developed by Ein in [Ein88; Ein91], where he proves, among other things, that if X is a general complete intersection of degree $(d_1, \ldots, d_k)$ of a smooth projective variety of dimension n, and $d_1 + \cdots + d_k \geq 2n - k + 1$, then all subvarieties of X are of general type. In view of Lang's Conjecture, this suggests a hyperbolicity conjecture for complete intersections:

Conjecture 1.5. *[Dem20] Let $X \subset \mathbb{P}^n$ be a complete intersection of hypersurfaces of respective degrees $d_1, \ldots, d_k$. If X is very general and $d_1 + \cdots + d_k \geq 2n - k$, then X is hyperbolic.*

By using jet differentials bundles, Brotbek [Bro14] was able to show that if $X \subset \mathbb{P}^n$ is a generic complete intersection of at least $n/2$ hypersurfaces of big enough degrees, then X is hyperbolic. Other partial results have also been found by Brotbek and Darondeau [BD18], Xie [Xie15], Deng [Den15; Den16], and Coskun and Riedl [CR20].

I believe the conjecture can be approached in its algebraic hyperbolic version by strengthening the techniques introduced by Clemens, Ein, and Voisin, and possibly by using algebraic jet bundles.

1.1.4. *Complements, pairs, and Green-Griffiths-Lang.* Another perspective on the hyperbolicity problem is over quasi-projective varieties, also called the log pair case. Here, we look at complements of divisors on a smooth projective variety and study the existence of entire curves on it. In $\mathbb{P}^n$, we expect a claim similar to the Kobayashi Conjecture to hold:

Conjecture 1.6. *[Kob70] The complement $\mathbb{P}^n \setminus D$ of a (very) general hypersurface $D \subset \mathbb{P}^n$ of degree $d \geq 2n + 1$ is hyperbolic, that is, every entire map $f : \mathbb{C} \to \mathbb{P}^n \setminus D$ is constant.*

This conjecture and its original version for hypersurfaces in $\mathbb{P}^n$ have been solved by Siu [Siu99; Siu02] for sufficiently high degree d. However, the currently known bounds are much larger than the conjectured bounds.

As a counterexample, if $d \leq 2n$, Zaidenberg [Zai87] showed that there exists a line intersecting a general degree $2n$ hypersurface D in two points, so $\mathbb{P}^n \setminus D$ is not hyperbolic.

More generally, let A be a smooth projective variety and D be a normal crossings divisor on A. We call (A, D) a log pair, and look at the complement $A \setminus D$. As in the projective case, hyperbolicity has an algebraic version: we say the pair (A, D) is algebraically hyperbolic if there exists an $\epsilon > 0$ such that for all nonconstant maps $f : C \to A$ from a smooth projective curve C to A, we have

$$2g(C) - 2 + |f^{-1}(D)| > \epsilon \deg f^* H$$

where H is an ample line bundle on A and $g(C)$ is the geometric genus of C. The conjectures of Demailly and Lang can be formulated in the log pair case as follows:

Conjecture 1.7. *[AT20] Let (A, D) be a log pair. Then the following are equivalent:*

- *$A \setminus D$ is hyperbolic.*
- *$A \setminus D$ is algebraically hyperbolic.*
- *The set of integral points of $A \setminus D$ is finite.*
- *All subvarieties of $A \setminus D$ are of log general type.*

Chen[Che04], Pacienza and Rousseau [PR07] proved that if $d \geq 2n + 1$ and $D \subset \mathbb{P}^n$ is a very general degree d hypersurface, then the complement $\mathbb{P}^n \setminus D$ is algebraically hyperbolic. This bound is sharp due to Zaiderberg's observation when $d \leq 2n$.

Similarly to Theorem 1.2, I expect to be able to extend these results to other varieties, such as complements of Grassmannians and toric varieties:

Question 1.8. For which degrees d is the complement of a very general hypersurface D of degree d of a Grassmannian $A = G(k, n)$ algebraically hyperbolic? What if A is a flag variety or a toric variety?

The Green-Griffiths-Lang Conjecture is also stated for varieties of log general type:

Conjecture 1.9. *Let (A, D) be a log pair such that $K_A + D$ is big. Then there is some proper subvariety $S \subsetneq A$ containing the images of all nonconstant holomorphic maps $\mathbb{C} \to A \setminus D$.*

This conjecture is still open in dimension two and above, even for complements of divisors in projective space. In [CRY22], Chen, Riedl, and Yeong prove the algebraic version for very general hypersurfaces $D \subset \mathbb{P}^n$ of degree $d = 2n$ and find the locus S when $n = 2$.

I plan to further understand the geometry of curves in varieties of log general type by improving the known results and contributing to the understanding of the Green-Griffiths-Lang Conjecture. For instance, what can we say about the locus S in $\mathbb{P}^n$? How about hypersurfaces in more general varieties?

2. Normal bundles of rational curves

The study of the space of rational curves has a history of celebrated results and important conjectures. Eisenbud, Van de Ven [EV81; EV82], Ghione and Sacchiero [GS80; Sac80; Sac82] give a stratification of the space of degree e rational curves in $\mathbb{P}^3$ by locally closed, irreducible subsets consisting of curves with a fixed splitting type for the normal bundle, and they show these strata have the expected dimension. Miret [Mir86] extends the analysis to $\mathbb{P}^n$, and defines a stratification of the variety of degree e rational curves in $\mathbb{P}^n$ in terms of the decomposition of the normal bundle. Ramella [Ram90; Ram93] shows that the locus of nondegenerate rational curves with a specified splitting type for the restricted tangent bundle $T_{\mathbb{P}^n}|_C$ is irreducible with the expected dimension.

However, the geometry of the space of rational curves in $\mathbb{P}^n$ with a given splitting type for the normal bundle can be more complicated. Alzati and Re [AR17] showed an example for which this locus is reducible. Then, Coskun and Riedl [CR18] show that this space has arbitrarily many components, and the difference between the expected dimension and the actual dimension of a component can grow arbitrarily large as the degree of the curve increases.

Rational curves over varieties represent an important tool in the development of higher-dimensional algebraic geometry as in [Deb01; Kol96]. The local structure of the space of rational curves C on a variety X is controlled by the normal bundle $N_{C/X}$. Let X be a degree d hypersurface in $\mathbb{P}^n$. For a given splitting type $E_{\vec{a}} = \bigoplus_{i=1}^{n-2} \mathcal{O}(a_i)$, we can consider the hypersurfaces $X = V(F)$ containing a fixed rational curve C:

$$\Sigma = \{F \mid X \text{ is a degree } d \text{ hypersurface smooth along } C\} \subset H^0(\mathcal{O}_{\mathbb{P}^n}(d))$$

and the subspace of those with the specified splitting type:

$$\Sigma_{\vec{a}} = \{F \in \Sigma \mid N_{C/X} \cong E_{\vec{a}}\} \subset \Sigma.$$

At this point, we raise two fundamental questions about $\Sigma_{\vec{a}}$:

- What are the splitting types achieved for some hypersurface X?
- Does $\Sigma_{\vec{a}}$ have the expected codimension in Σ?

The case of lines is done by Larson [Lar21], who shows that $\Sigma_{\vec{a}}$ is smooth of the expected dimension. Coskun and Riedl [CR19b] prove that the normal bundle on a degree $d \geq 2$ general hypersurface is balanced. In [Mio23b], I fully answer both questions stated above in the case of a rational normal curve C on a degree d (not necessarily general) projective hypersurface X. Throughout the proof, I develop a method to construct explicit hypersurfaces X for any given splitting type for the normal bundle, and I obtain a good description of $\Sigma_{\vec{a}}$ from the maps inducing $N_{C/X}$.

Theorem 2.1. *[Mio23b, Theorems 3.1 and 4.3] Let $e \leq n$, $d \geq 2$, and let C be the rational normal curve of degree e in $\mathbb{P}^n$. Then:*

(a) For all splitting types with at most $e-2$ summands of degrees greater than e, that is, of the form

$$E = \left(\bigoplus_{i=1}^{e-2} \mathcal{O}(e+2-a_i)\right) \oplus \left(\bigoplus_{j=e+1}^{n} \mathcal{O}(e-b_j)\right),$$

with $a_i, b_j \geq 0$ and $\sum_{i=1}^{e-2} a_i + \sum_{j=e+1}^{n} b_j = e(d-1)-2$, we obtain explicit examples of degree d hypersurfaces X, smooth along the curve C, with normal bundle $N_{C/X} \cong E$.

(b) *If $N_{C/X}$ is not of the form E above, then $e < n-1$, and the normal bundle must have the form*

$$E' = \mathcal{O}(e+2)^{e-1} \oplus \left(\bigoplus_{j=e+2}^{n} \mathcal{O}(e-b_j) \right),$$

with $b_j \geq 0$ and $\sum_{j=e+2}^{n} b_j = e(d-1)$. For each splitting type E', we find explicit examples of degree d hypersurfaces X, smooth along C, with normal bundle $N_{C/X} \cong E'$.

The proof relies on having a sharp control of the normal bundle sequence

$$0 \longrightarrow N_{C/X} \longrightarrow N_{C/\mathbb{P}^n} \xrightarrow{\psi} N_{X/\mathbb{P}^n}|_C \cong \mathcal{O}_{\mathbb{P}^1}(de)$$

and the maps ψ induced by hypersurfaces to create the appropriate ψ for each splitting type of the normal bundle.

Additionally, by describing the space of maps inducing $E_{\vec{a}}$, I showed that $\Sigma_{\vec{a}}$ is irreducible and smooth and computed its dimension. It is interesting to notice that when $e < n$ and there exist terms $a_i = e+2$, the expected and the actual codimension differ, and the difference between the actual and the expected codimension can get arbitrarily large as n grows.

Theorem 2.2. *[Mio23b, Theorem 3.6] The locus $\Sigma_{\vec{a}}$ is irreducible and smooth of codimension $h^1(\mathcal{E}nd(E_{\vec{a}})) - (n-e)z$ in Σ.*

When $d \geq 4$ and the field has characteristic 0, we can use the linear systems of hypersurfaces inducing a particular map ψ to obtain smooth hypersurfaces for each splitting type.

Theorem 2.3. *[Mio23b, Corollary 3.2] (char $K = 0$) Let $d \geq 4$, and assume the base field has characteristic 0. For all splitting types $E_{\vec{a}}$, there exists a smooth hypersurface X of degree d containing the curve C with normal bundle $N_{C/X} \cong E_{\vec{a}}$.*

The case of quadrics, $d = 2$, presents additional difficulties since not all maps $N_{C/\mathbb{P}^n} \xrightarrow{\psi} N_{X/\mathbb{P}^n}|_C$ are achieved [Mio23b, Proposition 4.1], and it is not clear what splitting types induce obtainable maps. However, by gaining a close understanding of the relation between the quadratic form matrix of X and the relations among the entries of ψ, I developed a method to modify the polynomial of X while keeping the same splitting type of $N_{C/X}$. This allowed me to get examples of higher ranks, and obtain a lower bound for the maximum rank of quadrics with a given normal bundle.

Theorem 2.4. *[Mio23b, Theorem 4.5]*

(a) *For every splitting type*

$$E = \left(\bigoplus_{i=1}^{e-2} \mathcal{O}(e+2-a_i) \right) \oplus \left(\bigoplus_{j=e+1}^{n} \mathcal{O}(e-b_j) \right),$$

with $a_i, b_j \geq 0$ and $\sum_{i=1}^{e-2} a_i + \sum_{j=e+1}^{n} b_j = e-2$, we obtain an example of quadric X of corank at most $\sum_{a_i \geq 4}(a_i - 3)$ with $N_{C/X} \cong E$. In particular, if $a_i \leq 3$ for all i, there exists a smooth quadric X with $N_{C/X} \cong E$.

(b) *For splitting types of the form*

$$E' = \mathcal{O}(e+2)^{e-1} \oplus \left(\bigoplus_{j=e+2}^{n} \mathcal{O}(e-b_j) \right),$$

with $b_j \geq 0$ and $\sum_{j=e+2}^{n} b_j = e$, let $w = |\{j \mid b_j = 0\}|$ be the number of terms of degree e. Then we obtain a quadric X of corank $e+1-\min\{e+1, n-e-w\}$ with $N_{C/X} \cong E'$. In particular, if $e+1 \leq n-e-w$, then there exists a smooth quadric X with $N_{C/X} \cong E'$.

2.1. **Further research problems.** I am interested in understanding the geometry of projective varieties via the study of the rational curves lying on them. In general, I seek to understand the space of rational curves and how they deform along the variety. Often, they can be characterized in terms of invariants such as the degrees of the hypersurface and the curve, and the positivity of the canonical bundle and the normal bundle of the curve over the hypersurface.

The immediate testing ground is degree d hypersurfaces X in projective space $\mathbb{P}^n$. When $d > 2n-3$, $n \geq 5$, we are in the algebraic hyperbolicity range and a very general X does not contain rational curves. When $d \leq n$, X is Fano, it contains many rational curves, and we can compute the dimension of the space of curves of a given degree. But if $n+1 < d \leq 2n-3$, we expect the degrees of the rational curves on X to be bounded, so for example, X might contain lines but not conics. In general, some leading questions we can ask are:

Question 2.5. For a general hypersurface X of degree d in $\mathbb{P}^n$, for which pairs (e, g) does there exist a curve of degree e and geometric genus g in X?

Question 2.6. As the degree of the hypersurface X changes, what happens to its spaces of rational curves?

Important work and precise conjectures have been made around these problems, such as the already mentioned Clemens' Conjecture, Debarre-de Jong Conjecture, and Voisin's Conjecture.

Here, I suggest some possible research directions, both building on my work on normal bundles of rational curves and also problems I want to approach in the future.

2.1.1. *Complete intersections.* In [CR19b, Proposition 3.5], Coskun and Riedl showed that if X is a complete intersection, and Y is a hypersurface of degree $d_k \geq 3$, both containing the rational normal curve C, then the normal bundle $N_{C/X\cap Y}$ is given as the kernel

$$0 \longrightarrow N_{C/X\cap Y} \longrightarrow N_{C/X} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d_k e) \longrightarrow 0.$$

From this, they were able to show that the normal bundle of a rational curve in a general complete intersection is balanced. We can also use this proposition to inductively compute the normal bundle of explicit (and non-general) complete intersections. We ask the same questions, now for complete intersections:

Question 2.7. Let X be a complete intersection containing a rational normal curve C. What are the possible splitting types for $N_{C/X}$? Can we obtain explicit examples of X for each possible splitting type?

In this case, however, I do not expect the space of complete intersections with a given splitting type for the normal bundle to be irreducible. But we can still ask about the dimension of the irreducible components.

Question 2.8. What is the dimension of the space of complete intersections with a given splitting type for the normal bundle? How many components can it have?

Another interesting approach would be to work directly with the normal bundle sequence of the $(d_1, \ldots, d_k)$ complete intersection X:

$$0 \longrightarrow N_{C/X} \longrightarrow N_{C/\mathbb{P}^n} \xrightarrow{\psi} \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^1}(d_i e) \longrightarrow 0.$$

By combining Theorem 3.1 and Proposition 3.5 of [CR19b] when all $d_i \geq 3$, we get that all maps ψ above are achieved as we vary X. But what maps correspond to a fixed splitting type? We can translate it to a question purely about kernel bundles:

Question 2.9. What are the maps $\psi : \mathcal{O}_{\mathbb{P}^1}(e+2)^{e-1} \oplus \mathcal{O}_{\mathbb{P}^1}(e)^{n-e} \to \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^1}(d_i e)$ with a given kernel $E \cong \bigoplus_{i=1}^{n-k-1} \mathcal{O}_{\mathbb{P}^1}(a_i)$?

2.1.2. *Quadrics case.* The dimension of $\Sigma_{\vec{a}}$ in the case of quadrics remained open, as it is not clear what maps ψ come from some quadric hypersurface X.

Question 2.10. Can we describe what maps $N_{C/\mathbb{P}^n} \xrightarrow{\psi} N_{X/\mathbb{P}^n}|_C$ appear when X is a quadric hypersurface? What is the dimension of $\Sigma_{\vec{a}}$ in this case?

This case is open even for general complete intersections, where Coskun and Riedl conjectured the normal bundle to be balanced:

Conjecture 2.11. *[CR19b, Conjecture 4.6] Let X be a general Fano complete intersection of k quadric hypersurfaces in $\mathbb{P}^n$ containing the degree n rational normal curve of degree R_n. Then $N_{R_n/X}$ is balanced.*

By varying the hypersurface X, we vary the map ψ, and get a morphism

$$\phi : H^0(\mathcal{I}_C(d)) \to \mathrm{Hom}(N_{C/\mathbb{P}^n}, \mathcal{O}(de))$$

with kernel $H^0(\mathcal{I}_C^2(d))$. It is surjective for $d \geq 3$ [CR19b, Theorem 3.1]. For $d = 2$ and the rational normal curve C of degree n, ϕ is injective:

Proposition 2.12. *[Mio23b, Proposition 4.1] The dimension of $H^0(\mathcal{I}_C^2(2))$ is $\frac{(n-e)(n-e+1)}{2}$. In particular, ϕ is injective when $e = n$.*

I expect to be able to better describe the image of ϕ. A good picture of it in the hypersurface case suffices to obtain an answer to Conjecture 2.11 and to compute the dimension of $\Sigma_{\vec{a}}$ in the quadric case.

2.1.3. *Restricted tangent bundle.* The restriction of the tangent bundle of X to the curve, $T_X|_C$, is another important tool in the deformation theory of C on X. The splitting type of the restricted tangent bundle has been studied by many authors for curves in $\mathbb{P}^n$ [Asc88; Ram90; Ram93; Ran01; AI13; AR15; AI16; Asc22; LR23] and other varieties [Asc86; Man21; BS23].

I believe I can extend the methods used for the normal bundle to obtain similar results for the restricted tangent bundle of a curve on a hypersurface. We ask:

Question 2.13. Let X be a degree d hypersurface in $\mathbb{P}^n$ containing a rational curve C of degree e.

- If X is general, then is $T_X|_C$ balanced?
- What are the possible splitting types of $T_X|_C$? Are all of them achieved for smooth hypersurfaces?
- What is the dimension of the space of hypersurfaces X such that $T_X|_C$ has a given splitting type? We expect it to have codimension $h^1(\mathbb{P}^1, \mathcal{E}nd(T_X|_C))$.

2.1.4. *Spaces of curves on hypersurfaces.* Let X be a general degree d hypersurface in $\mathbb{P}^n$. Riedl and Yang [RY19] proved that when $n \geq d+2$, the Kontsevich space $\overline{\mathcal{M}}_{0,0}(X, e)$ of degree e rational curves in X is irreducible of dimension $e(n-d+1)+n-4$, almost completely solving the Coskun-Harris-Starr Conjecture for Fano hypersurfaces.

As already mentioned, celebrated results by Eisenbud, Van de Ven [EV81; EV82], Ghione and Sacchiero [GS80; Sac80; Sac82], and Miret [Mir86] give a stratification of the space of degree e rational curves in $\mathbb{P}^n$ in terms of their normal bundle. However, for $n \geq 5$, Alzati and Re [AR17], show an example where the space of curves with a given splitting type is reducible. Coskun and Riedl [CR18] show they do not need to have the expected dimension.

I question what we can say about the space of rational curves on a hypersurface in terms of the splitting type of the normal and tangent bundle.

Question 2.14. Let X be a degree d hypersurface in $\mathbb{P}^n$, and let $M_e(X)(E)$ be the space of degree e rational curves C in X whose normal bundles $N_{C/X}$ have a given splitting type $E \cong \bigoplus_{i=1}^{n-2} \mathcal{O}(a_i)$.

- How many irreducible components does $M_e(X)(E)$ have?

- Does the general curve have balanced splitting type?
- What are the dimensions of the components of $M_e(X)(E)$? Is it the expected dimension?
- Can we define a stratification of the space of rational curves by spaces $M_e(X)(E)$?

We can also return to our perspective with a fixed rational curve, and ask if there is a stratification in this case:

Question 2.15. Let C be a rational curve in $\mathbb{P}^n$, and let Σ be the space of all degree d hypersurfaces X containing C. Can we define a stratification of Σ in terms of the splitting type of the normal bundle $N_{C/X}$?

We can also ask the questions above in terms of the restricted tangent bundle $T_X|_C$. Differently from the normal bundle case, Ramella [Ram90; Ram93] showed that the locus of rational curves in $\mathbb{P}^n$ with a given splitting type for the restricted tangent bundle is irreducible with the expected dimension. Is it also nicely behaved in the case of curves on a hypersurface X?

2.1.5. *Curves on hypersurfaces of general type and Voisin's Conjecture.* As we increase the degree d of the very general hypersurface $X \subset \mathbb{P}^n$ it stops containing rational curves. When $d > 2n-3$, X does not contain any rational curves as we discussed about algebraic hyperbolicity. But what rational curves can X contain when $n+1 < d \leq 2n-3$, that is, when X is of general type? Voisin conjectured [Voi03] the degrees of rational curves on a very general hypersurface of general type are bounded. If the general Coskun-Harris-Starr Conjecture is true, Voisin's conjecture follows, and we get the exact bound for the degree:

Conjecture 2.16. *A (very) general hypersurface X of degree d in $\mathbb{P}^n$, $(d,n) \neq (4,3)$, contains rational curves of degree d if and only if*

$$d > \frac{e+1}{e}n + \frac{e-4}{e}.$$

For example:

- If $\frac{3}{2}n - 1 < d \leq 2n-3$, then X contains lines but no other rational curves.
- If $\frac{4}{3}n - \frac{1}{3} < d \leq \frac{3}{2}n - 1$, then X contains lines and conics, but no other rational curves.
- If $\frac{5}{4}n < d \leq \frac{4}{3}n - \frac{1}{3}$, then X contains lines, conics, and cubics, but no other rational curves.

Riedl and Yang [RY20] show that if $\frac{3n+1}{2} \leq d \leq 2n-3$, then X contains lines but no other rational curves. The proof relies on the techniques build up from Clemens, Ein, Voisin, Pacienza, and Ran, which consist in using the Lazarsfeld-Mukai bundle of hyperplanes M_H to show that through a general point z of a rational curve Z of X there exists a line $l(z)$. By controlling how the line intersects X, they show that $l(z)$ is actually contained in X. Later, Coskun and Riedl [CR22] obtain the optimal bound $d \geq \frac{3n}{2}$ by using clustered families of subspaces in Grassmannians.

Abe [Abe23] follows this approach considering linear systems of quadrics instead of hyperplanes, and proves that if $d > \frac{7}{5}n + \frac{16}{5}$, then any rational curve on X is a line or a conic.

The methods used in this conjecture proved to be powerful, and involve techniques I learned both from working on rational curves and on algebraic hyperbolicity. My goal is to enhance these methods and apply them to make progress on Voisin's conjecture, as well as approach problems such as the Lang conjectures.

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